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## *Dielectric Properties of Biological Materials*

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### 3.1 Introduction

At some level of organization, all matter consists of charged entities held together by various atomic, molecular, and intermolecular forces. The effect of an externally applied electric field on the charge distribution is specific to the material; the dielectric properties are a measure of that effect; they are intrinsic properties of matter used to characterize nonmetallic materials. Biological matter has free and bound charges; an applied electric field will cause them to drift and displace, thus inducing conduction and polarization currents. Dielectric spectroscopy is the science that relates the dielectric properties to the underlying microscopic mechanisms of polarization and conduction. These dielectric phenomena are determined by and are informative about the structure and composition of the material. Consequently, knowledge of the dielectric properties is of practical importance in all fields of science where electromagnetic fields impinge or are used to probe or process matter. It is equally important in biomedical fields such as electrophysiology, where endogenous bioelectric sources provide signals that are sensed through various body tissues and are affected by their dielectric properties.

The past decade has seen a dramatic increase in the exposure of people to electromagnetic fields from wireless telecommunication devices and infrastructure. This situation sparked large research programs on the assessment and quantification of exposure of people and on the biological effects resulting from the exposure. Information on the dielectric properties of tissues is vital to these studies, for the computation of exposure metrics and the provision of a mechanistic explanation for biological effects. To satisfy the need of current research activity, this chapter will review the dielectric data for body tissue and the underlying mechanisms of interaction at the cellular, subcellular, and molecular levels. In doing so, we will draw on the authoritative article by Foster and Schwan (1996), published in the second edition of this book, which goes a long way toward establishing dielectric spectroscopy as a powerful tool for mechanistic studies.

Another area of scientific activity in the last decade revolved around the formulation of standard procedures for the experimental assessment of human exposure from electromagnetic sources, mostly telecommunication radio transceivers and their accessories. This created the need to formulate and measure the dielectric properties of tissue equivalent material and made dielectric measurement and the assessment of the associated uncertainty part of the compliance testing procedure. This chapter will deal with the fundamental issues that need to be established if dielectric measurement is to become a routine but accurate laboratory procedure.

It is not possible to review the dielectric properties and the polarization mechanisms in biological material without singling out the contribution to this field of Herman P. Schwan, whose name is associated with all the major findings over the past 50 years (Foster, 2002). Indeed, his 1957 review of the bulk electrical properties of cells and cell suspensions is one of the earliest and most studied texts on the subject. There are other reviews by Pethig (1979), Stuchly (1979), Schwan and Foster (1980), Pethig and Kell (1987), and Foster and Schwan (1989). The fundamental aspects and a more extensive treatment of the theoretical aspects of this subject can be found in books by Cole (1972), Grant et al. (1978), Schanne and Ceretti (1978), Pethig (1979), and more recently, Craig (1995), Roussy and Pearce (1995), and Grimnes and Martinsen (2000).

For most biological materials, the magnetic permeability is close to that of free space (i.e., diamagnetic), which implies that there is no direct interaction with the magnetic component of electromagnetic fields at low field strengths. However, this position is now changing following relatively recent reports of the presence of magnetite in human nervous tissue (see, e.g., Dobson and Grassi, 1996), which suggest that magnetite may provide a mechanism for direct interaction of external magnetic fields with the human central nervous system. The role of these strongly magnetic materials in organisms is only just beginning to be unraveled. This subject is elaborated in the second part of next chapter.

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## 3.2 Dielectric Properties—Molecular Origin

From a historical perspective, the dielectric properties of materials were first observed experimentally by Faraday in the 1830s as a change in the capacity of an empty capacitor when a material is introduced inside it. Faraday introduced the term *specific inductive capacity* to describe the ratio of the capacities of the filled and empty capacitor. This quantity is now known as the permittivity and is denoted by  $\epsilon$ . It is a fundamental property of nonmetallic or dielectric materials. Under quasistatic conditions, the capacitance  $C_0$  of a perfect capacitor of area  $A$  and plate separation  $d$  changes to  $C > C_0$ :

$$C = \epsilon \epsilon_0 = \epsilon \frac{\epsilon_0 A}{d} \quad (3.1)$$

where  $\epsilon$  is the relative permittivity of the material (dimensionless),  $\epsilon_0$  is the permittivity of free space ( $8.8542 \times 10^{-12}$  F/m), and the product  $\epsilon \epsilon_0$  is the absolute permittivity. The increase in capacity is due to the additional charge density induced by the field in the material. The field is said to have polarized the medium; polarizability or the ability of the material to polarize is the main determinant of its dielectric properties.

This section will start with the fundamental concepts of the interaction of homogenous matter with static fields and proceed, in steps, to heterogenous mixtures and their dynamic response to time-varying fields leading to the dielectric properties of tissues.

### 3.2.1 Quasi-Static Response

Considering the simple case of a monomolecular material, three main interaction mechanisms are possible: electronic, atomic, and molecular polarization. Electronic polarization is the shift of electrons, in the direction of the field, from their equilibrium position with respect to the positive nuclei. Atomic polarization is the relative displacement of

atoms or atom groups relative to each other. The orientation of permanent or induced molecular dipoles, when present, is known as molecular polarization. The total polarizability  $\alpha_T$  is the sum of the contribution of, in this case, all three processes, termed  $\alpha_e$ ,  $\alpha_a$ , and  $\alpha_d$ .

In view of the relative importance of molecular orientation processes in defining the total polarization ( $\alpha_e + \alpha_a < \alpha_d$ ) and hence the dielectric properties, it is usual to differentiate between the polar and nonpolar materials when these properties are considered.

For ideal nonpolar materials, the relationship  $\varepsilon = n^2$ , where  $n$  is the optical refractive index, holds true. When a dielectric material becomes polarized by the application of an external electric field  $E$ , the dipole moment of the constituent molecules is given by

$$\bar{m} = \alpha_T \bar{E}_1 \quad (3.2)$$

where  $E_1$  is the local field acting on the molecules. The dipole moment per unit volume of the material  $P$  increases the total displacement flux density  $D$ , defined from the relationship  $D = \varepsilon_0 E$  in vacuum and  $D = \varepsilon_0 \varepsilon E$  in a medium of relative permittivity  $\varepsilon$ . The latter expression may also be written as

$$D = \varepsilon_0 E + P \quad (3.3)$$

The dependence of  $P$  on  $E$  can take several forms, the simplest and most common being a scalar proportionality:

$$P = \varepsilon_0 \chi E \quad (3.4)$$

where  $\chi = \varepsilon - 1$  is the relative dielectric susceptibility. This simple relationship is valid for a perfect isotropic dielectric, at low or moderate field intensities and at static or quasi-static field frequencies.

If the material contains  $N$  dipoles per unit volume, then

$$P = N \alpha_T E_1 \quad (3.5)$$

and

$$\varepsilon - 1 = \frac{N \alpha_T}{\varepsilon_0} \frac{E_1}{E} \quad (3.6)$$

The molecular description of the permittivity requires that the relationship between the microscopic and the macroscopic field intensities be known. In most cases, there is no exact solution to this problem, only more or less good approximations that hold within the confines of the assumptions and simplifications made, as will be briefly illustrated for typical classes of materials.

### 3.2.2 Permittivity of Low-Pressure Gases

At low pressures the molecules are far apart from each other, and their interaction with each other may be assumed to be negligible in comparison with the macroscopic field intensity  $E$ . Under these conditions  $E_1 \approx E$  and

$$\varepsilon - 1 = \frac{N \alpha_T}{\varepsilon_0} \quad (3.7)$$

The relative permittivity of a nonpolar gas is very close to 1, typically of the order of 1.0001 at atmospheric pressure.

### 3.2.3 Permittivity of Liquids and Dense Gases

When the intermolecular interactions are such that  $E_1 \neq E$ , the local field must be estimated. One approach is to consider a spherical region inside the dielectric that is large compared to the size of a molecule; the field inside it is estimated for nonpolar materials to be

$$E_1 = \left( \frac{\varepsilon + 2}{3} \right) E \quad (3.8)$$

which yields

$$\frac{3(\varepsilon - 1)}{\varepsilon + 2} = \frac{N}{\varepsilon_0} \alpha_T \quad (3.9)$$

The above expression is known as the Claussius–Mossoti–Lorentz formulation. It is not always valid, such as when the density of the material corresponds to  $N = 3 \varepsilon_0 / \alpha_T$ . An alternative formulation, valid when the molecules are polarizable point dipoles of permanent moment  $\mu$ , was provided by Onsager:

$$\frac{(\varepsilon - n^2)(2\varepsilon + n^2)}{\varepsilon(n^2 + 2)^2} = \frac{N\mu^2}{9kT\varepsilon_0} \quad (3.10)$$

where  $k$  is the Boltzman constant and  $T$  is the absolute temperature.

Debye separated out the contribution to the total polarization of the permanent dipole from those associated with electronic and atomic displacements and arrived at the following relationship

$$\frac{\varepsilon - 1}{\varepsilon + 2} = \frac{N}{3\varepsilon_0} (\alpha + \mu^2/3kT) \quad (3.11)$$

While Onsager and Debye used semistatistical techniques to estimate the local field, others like Kirkwood, and later Fröhlich, used statistical methods to obtain a rigorous expression of the permittivity after taking local interactions into consideration and obtained

$$\frac{(\varepsilon - 1)(2\varepsilon + 1)}{3\varepsilon} = \frac{N}{\varepsilon_0} (\alpha + g\mu^2/3kT) \quad (3.12)$$

This is Kirkwood's equation for permittivity in which  $g$  is known as the Kirkwood correlation parameter, introduced to account for the effect of local ordering in the material. Fröhlich's theory gives

$$\frac{(\varepsilon - n^2)(2\varepsilon + n^2)}{\varepsilon(n^2 + 2)^2} = \frac{Ng\mu^2}{9kT\varepsilon_0} \quad (3.13)$$

which, except for the correlation parameter  $g$ , is identical to Onsager's equation.

This brief outline of the dielectric theory gives an idea of the nature of the electric field interaction problems and of the various techniques used to partially solve them under static field conditions. The solutions hold for slow time-varying fields as long as there is a quasistatic state. References to the original work by Debye (1929), Kirkwood (1936), Onsager (1936), and Fröhlich (1955) are given in Böttcher and Bordewijk (1978) and other well-known texts (Hill et al., 1969; Jonscher, 1983).

### 3.3 Time and Frequency Dependence of the Dielectric Response

Much of the interest in the dielectric properties of biological materials is concerned with their response to time-varying electric fields. This can be explained by the same macroscopic variables used for the quasi-static state except for the introduction of a time dependence for the excitation and response. The general discussion will assume sinusoidal fields and linear and isotropic responses, nonsinusoidal fields and material anisotropy, and nonlinearity being special cases.

#### 3.3.1 Time-Dependent Polarization—Impulse Response—Kramers–Krönig Relations

The following relationship holds irrespective of the polarization mechanism:

$$P(t) = D(t) - \epsilon_0 E(t) \quad (3.14)$$

For an ideal dielectric material with no free charge, the polarization follows the pulse with a delay determined by the time constant of the polarization mechanism. Assuming a rate process, which is that the rate of polarization is proportional to the constantly decreasing number of unpolarized units, the simplest expression for the polarization is obtained from the solution of the first-order differential rate equation with constant coefficients and time constant  $\tau$ , giving

$$P(t) = P(1 - e^{-t/\tau}) \quad (3.15)$$

The decay of polarization is also an exponential function

$$P(t) = e^{-t/\tau} \quad (3.16)$$

For a linear system, the response to a unit-step electric field is the impulse response  $f(t)$  of the system. The response of the system to a time-dependent field can be obtained from summation in a convolution integral of the impulses corresponding to a sequence of elements making up the electric field. For a harmonic field and a causal, time-independent system, the Fourier transform exists and yields

$$P(\omega) = \epsilon_0 \chi(\omega) E(\omega) \quad (3.17)$$

indicating that the dielectric susceptibility  $\chi(\omega)$  is the Fourier transform of  $f(t)$ . In general, the susceptibility is a complex function reflecting the fact that it informs on the magnitude and phase of the polarization with respect to the polarizing field

$$\chi(\omega) = \chi' - j\chi'' \quad (3.18)$$

The real and imaginary parts of  $\chi(\omega)$  can be obtained from the separate parts of the Fourier transform:

$$\begin{aligned}\chi'(\omega) &= \int_{-\infty}^{+\infty} f(t) \cos(\omega t) dt = \int_0^{+\infty} f(t) \cos(\omega t) dt \\ \chi''(\omega) &= \int_{-\infty}^{+\infty} f(t) \sin(\omega t) dt = \int_0^{+\infty} f(t) \sin(\omega t) dt\end{aligned}\quad (3.19)$$

The limit of integration can be changed from  $-\infty$  to 0 since  $f(t)$  is causal.

The impulse response  $f(t)$  defines the dielectric response, and conversely, knowledge of the complex susceptibility allows the determination of the impulse response by carrying out the reverse transformation, which gives  $f(t)$  in terms of either  $\chi'(\omega)$  or  $\chi''(\omega)$ :

$$\begin{aligned}f(t) &= (2/\pi) \int_0^{+\infty} \chi'(\omega) \cos(\omega t) d\omega \\ f(t) &= (2/\pi) \int_0^{+\infty} \chi''(\omega) \sin(\omega t) d\omega\end{aligned}\quad (3.20)$$

Eliminating  $f(t)$  from the above equations gives an expression of  $\chi'(\omega)$  in terms of  $\chi''(\omega)$  and vice versa. Thus, there is a relationship between the real and imaginary parts of the complex susceptibility of any material, such that knowledge of either enables the other to be calculated. The expressions of real and imaginary parts of the susceptibility or permittivity in terms of each other are known as the Kramers–Krönig relations and have been derived as

$$\begin{aligned}\chi'(\omega) &= \chi'(\infty) + \frac{2}{\pi} \int_0^{\infty} \frac{u\chi''(u) - \omega\chi''(\omega)}{u^2 - \omega^2} du \\ \chi''(\omega) &= \frac{2}{\pi} \int_0^{\infty} \frac{\chi'(u) - \chi'(\omega)}{u^2 - \omega^2} du\end{aligned}\quad (3.21)$$

where  $u$  is a variable of integration. Recalling the relationship between relative dielectric susceptibility and relative permittivity  $\chi = \varepsilon - 1$ , the permittivity is a complex function given by

$$\hat{\varepsilon}(\omega) = \varepsilon'(\omega) - j\varepsilon''(\omega) = (1 + \chi'(\omega) - j\chi''(\omega))\quad (3.22)$$

Thus, the Kramers–Krönig relations relate  $\varepsilon'$  to the complete spectrum of  $\varepsilon''$  and vice versa. A clear account of their derivation can be found in Jonscher (1983).

### 3.3.2 Permittivity of a Polar Substance—The Debye Equation

When a step field  $E$  is applied to a polar dielectric material, the electronic and atomic polarizations are established almost instantaneously compared to the time scale of the

molecular orientation; the total polarization reaches a steady state as a first-order process characterized by the time constant of the dipolar rotation. When the field is removed, the process is reversed; electronic and atomic polarizations subside first, followed by a relatively slow decay in dipolar polarization (Figure 3.1). The time constant  $\tau$  depends on the physical process, in this case the rotational dynamics of the dipole determined by the size, shape, and intermolecular relations of the molecules. If  $P_\infty$  and  $P_0$  are the instantaneous and steady-state polarization, respectively, then the total polarization for a first-order process characterized by a time constant  $\tau$  is

$$P = P_\infty + (P_0 - P_\infty)(1 - e^{-t/\tau}) \quad (3.23)$$

In time-varying fields the permittivity is a complex function originating from the magnitude and phase shift of the polarization with respect to the polarizing field:

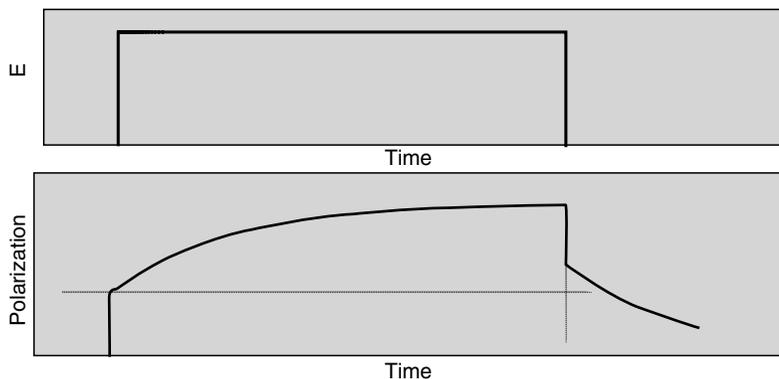
$$\hat{\epsilon} = \epsilon' - j\epsilon'' = \epsilon' - j\sigma/\omega\epsilon_0 \quad (3.24)$$

The real part  $\epsilon'$  is a measure of the induced polarization per unit field and the imaginary part  $\epsilon''$  is the out-of-phase loss factor associated with it. The loss factor can also be represented by a conductivity term  $\sigma = \omega\epsilon_0\epsilon''$  where  $\omega$  is the angular frequency. The SI unit of conductivity is siemens per meter (S/m).

The frequency response of the first-order system is obtained from the Laplace transformation, which provides the relationship known as the Debye equation:

$$\hat{\epsilon} = \epsilon_\infty + \frac{(\epsilon_0 - \epsilon_\infty)}{1 + j\omega\tau} = \epsilon' - j\epsilon'' \quad (3.25)$$

The limiting values of the permittivity,  $\epsilon_s$  and  $\epsilon_\infty$ , are known as static and infinite permittivity, respectively. The relaxation time  $\tau$  corresponds to a relaxation frequency  $f_r = 1/2\pi\tau$ . For a highly associated liquid such as water, the static permittivity can be expressed in terms of molecular parameters in accordance with the discussions in the previous section as



**FIGURE 3.1**

Schematic graph of dielectric response of a polar material expressed as polarization in the time domain in response to the step onset and removal of a polarizing electric field  $E$ . The nearly instantaneous rise in polarization, indicated by the horizontal dotted line, is  $P_\infty$ ; the vertical dotted line marks the instant and equivalent drop in polarization occurs when the field is removed.

$$\epsilon_s = \epsilon_\infty + \frac{Ng\mu^2}{2kT\epsilon_0} \quad (3.26)$$

The relaxation time may be identified with the time constant of the molecular polarization and expressed in terms of molecular parameters. If  $\eta$  is the viscosity, then for a spherical molecule of radius  $a$

$$\tau = 4\pi a^3 \eta / kT \quad (3.27)$$

For most polar materials, though not for water,  $\epsilon_\infty$  corresponds to the optical permittivity and is equal to the square of optical refractive index  $n$  of the medium:

$$\epsilon_\infty = n^2 \quad (3.28)$$

The dielectric properties of polar molecules vary with temperature; in general, both  $\epsilon_s$  and  $\tau$  decrease with increasing temperature.

As with the charge density and polarization, the time dependence of the current density  $J$  and  $\sigma$ , the current density per unit field, also follows a first-order law such that

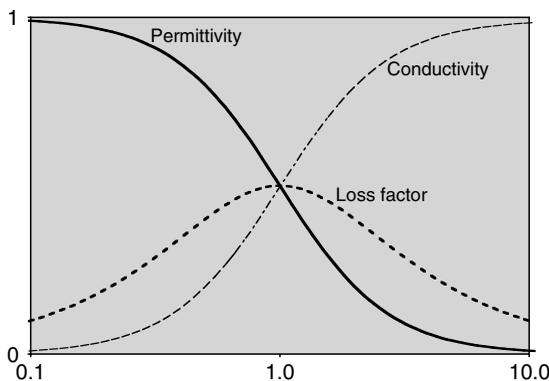
$$J/E = \sigma_\infty + (\sigma_s - \sigma_\infty)(1 - e^{-t/\tau}) \quad (3.29)$$

This transforms into the conductivity equivalent of the Debye equation:

$$\hat{\sigma} = \sigma_\infty + \frac{(\sigma_s - \sigma_\infty)}{1 + j\omega\tau} \quad (3.30)$$

Figure 3.2 shows the variation in the permittivity, loss factor, and conductivity with frequency for a single time constant relaxation; such behavior pertains to an idealized monomolecular polar substance with no residual frequency-independent conductivity, that is  $\sigma_s = 0$ . The best, if not the only, example of such material is pure water as will be discussed later.

At the relaxation frequency, the permittivity is halfway between its limiting values and the loss factor at its highest. In the case of a single time constant as described in Figure 3.2, the conductivity is halfway between its limiting values at the relaxation frequency.



**FIGURE 3.2** Normalized permittivity  $(\epsilon' - \epsilon_\infty)/(\epsilon_s - \epsilon_\infty)$ , loss factor  $\epsilon''/(\epsilon_s - \epsilon_\infty)$ , and conductivity  $\omega\epsilon_0 \epsilon''/(\epsilon_s - \epsilon_\infty)$  for a single time constant relaxation plotted against  $f/f_r$ .

### 3.3.3 Nonpolar Molecules

The permittivity of nonpolar materials is virtually constant throughout the frequency range. In general, the temperature dependence is not significant. The static and optical values of the permittivity are almost identical, hence the Maxwell relation  $\varepsilon = n^2$  holds true throughout the frequency and temperature range.

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## 3.4 Observed Responses of Real Systems—Conduction—Multiple Relaxations—The Universal Law

In the previous section we described the expected behavior of idealized materials; we now need to deal with the observed responses of real systems. Few materials exhibit single relaxation time dispersions as in the Debye model; real materials depart from this ideal behavior to a greater or lesser extent depending on the complexity of the underlying mechanisms. To describe these responses we need to introduce the concepts of multiple dispersions and distribution of relaxation time. Moreover, biological materials exhibit conduction as well as polarization mechanisms; this needs to be taken into consideration in describing their dielectric response.

### 3.4.1 Conduction

The Debye expression does not include the effect of conduction currents as would arise from, for example, the drift of free ions in static fields. If  $\sigma_s$  is the static conductivity, the Debye expression becomes

$$\hat{\varepsilon} = \varepsilon_\infty + \frac{(\varepsilon_s - \varepsilon_\infty)}{1 - j\omega\tau} - \frac{j\sigma_s}{\omega\varepsilon_0} \quad (3.31)$$

In terms of real and imaginary parts we have

$$\begin{aligned} \varepsilon' &= \varepsilon_\infty + \frac{(\varepsilon_s - \varepsilon_\infty)}{1 + (\omega\tau)^2} \\ \varepsilon'' &= \frac{\sigma_s}{\omega\varepsilon_0} + \frac{(\varepsilon_s - \varepsilon_\infty)\omega\tau}{1 + (\omega\tau)^2} \end{aligned} \quad (3.32)$$

The total conductivity  $\sigma$  is given by

$$\sigma = \omega\varepsilon_0\varepsilon'' = \sigma_s + \frac{(\varepsilon_s - \varepsilon_\infty)\varepsilon_0\omega^2\tau}{1 + (\omega\tau)^2} \quad (3.33)$$

The total conductivity is thus made of two terms corresponding to the residual static conductivity and polarization losses. In practice, it is only possible to measure the total conductivity of a material;  $\sigma_s$  is obtained from data analysis or by measurement at frequencies corresponding to  $\omega\tau \ll 1$  where the dipolar contribution to the total conductivity is negligible.

### 3.4.2 Multiple Relaxation Models—Distribution of Relaxation Times—Fractional Power Law Behavior

The occurrence of multiple interaction processes or the presence of more than one molecular conformational state or type of polar molecule may cause the dielectric behavior of a substance to exhibit multiple relaxation time dispersions. Deviation from Debye behavior may also indicate a polarization process whose kinetics are not first order or the presence of a complex intermolecular interaction. Models are needed to analyze the dielectric spectra of complex systems to unravel the underlying interaction mechanisms.

The simplest case is that of a dielectric response arising from multiple first-order processes; in this case the dielectric response will consist of multiple Debye terms to correspond to the polarization processes such that

$$\hat{\epsilon} = \epsilon_{\infty} + \frac{\Delta\epsilon_1}{1 - j\omega\tau_1} + \frac{\Delta\epsilon_2}{1 - j\omega\tau_2} + \dots \quad (3.34)$$

where  $\Delta\epsilon_n$  corresponds to the limits of the dispersion characterized by time constant  $\tau_n$ . If the relaxation times are well separated such that  $\tau_1 \ll \tau_2 \ll \tau_3 \ll \dots$ , a plot of the dielectric properties as a function of frequency will exhibit clearly resolved dispersion regions.

If, as is quite often the case, the relaxation times are not well separated, the material will exhibit a broad dispersion encompassing all the relaxation times. In the limit of a continuous distribution of relaxation times, the multiple Debye expression would be

$$\hat{\epsilon} = \epsilon_{\infty} + (\epsilon_s - \epsilon_{\infty}) \int_0^{\infty} \frac{\rho(\tau) d\tau}{1 - j\omega\tau} \quad (3.35)$$

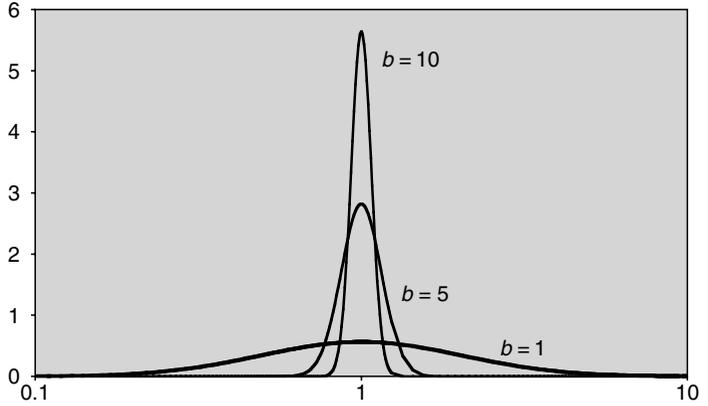
where

$$\int_0^{\infty} \rho(\tau) d\tau = 1 \quad (3.36)$$

The above equations can be used to represent all dielectric dispersion data, provided an appropriate distribution function  $\rho(\tau)$  is available. Conversely, it should also be possible, at least in principle, to invert dielectric relaxation spectra to determine  $\rho(\tau)$  directly; however, this is not easily achievable in practice. More commonly, one has to assume a distribution to describe the frequency dependence of the dielectric properties observed experimentally. The choice of distribution function should depend on the cause of the multiple dispersions in the material. For example, one can assume a Gaussian distribution as is known to occur for other physical characteristics (Figure 3.3) would be

$$\rho(t/\tau) = \frac{b}{\sqrt{\pi}} e^{-b^2[\ln(t/\tau)]^2} \quad (3.37)$$

where  $\tau$  is the mean relaxation time. The shape of the Gaussian function depends on the parameter  $b$ ; it reduces to the delta function when  $b$  tends to infinity and becomes very broad when  $b$  decreases; the area under the curve remains the same as required by the normalization condition. Incorporated into the expression for complex permittivity, it produces an expression that cannot be solved analytically, which makes it impractical for experimental data analysis.



**FIGURE 3.3**  
Gaussian distribution function as a function of  $t/\tau$ .

Numerous empirical distribution functions or models have been proposed to model the experimental data without elaboration of the underlying mechanisms. One of the most commonly used models, a modified version of the Debye expression, was proposed in 1941 by Cole and Cole and is widely known as the Cole–Cole model:

$$\hat{\epsilon} = \epsilon_{\infty} + \frac{(\epsilon_s - \epsilon_{\infty})}{1 - (j\omega\tau)^{1-\alpha}} = \epsilon' - j\epsilon'' \quad (3.38)$$

In it,  $\alpha$  is a distribution parameter in the range  $1 > \alpha \geq 0$ ; for  $\alpha = 0$ , the model reverts to the Debye equation. The real and imaginary parts are

$$\begin{aligned} \epsilon' &= \epsilon_{\infty} + \frac{(\epsilon_s - \epsilon_{\infty})[1 - (\omega\tau)^{1-\alpha} \sin(\alpha\pi/2)]}{1 + (\omega\tau)^{2(1-\alpha)} + 2(\omega\tau)^{1-\alpha} \sin(\alpha\pi/2)} \\ \epsilon'' &= \frac{(\epsilon_s - \epsilon_{\infty})(\omega\tau)^{1-\alpha} \cos(\alpha\pi/2)}{1 + (\omega\tau)^{2(1-\alpha)} + 2(\omega\tau)^{1-\alpha} \sin(\alpha\pi/2)} \end{aligned} \quad (3.39)$$

Eliminating  $\omega\tau$  from the above equations gives

$$\left(\epsilon' - \frac{(\epsilon_s + \epsilon_{\infty})}{2}\right)^2 + \left(\epsilon'' + \frac{(\epsilon_s + \epsilon_{\infty})}{2} \cot \frac{(1-\alpha)\pi}{2}\right)^2 = \left(\frac{\epsilon_s - \epsilon_{\infty}}{2} \operatorname{cosec} \frac{(1-\alpha)\pi}{2}\right)^2 \quad (3.40)$$

indicating that a plot of  $\epsilon'$  against  $\epsilon''$  is a semicircle with its center below the real axis. For  $\alpha = 0$ , the Debye equivalent of the above equation is

$$\left(\epsilon' - \frac{(\epsilon_s + \epsilon_{\infty})}{2}\right)^2 + \epsilon''^2 = \left(\frac{\epsilon_s - \epsilon_{\infty}}{2}\right)^2 \quad (3.41)$$

which indicates that  $\epsilon'$  against  $\epsilon''$  is a semicircle with its center on the real axis (Figure 3.4); these semicircle plots are known as Cole–Coles.

The distribution function that corresponds to the Cole–Cole model is

$$\rho(t/\tau) = \frac{1}{2\pi} \frac{\sin(\alpha\pi)}{\cos h[(1-\alpha) \ln(t/\tau)] - \cos(\alpha\pi)} \quad (3.42)$$